VIII. RADIATION IN THE EARLY UNIVERSE

Up to now we have ignored radiation. At early time (*i.e.*, large z), radiation dominates the energy density of the universe. We wish to calculate both the time (or redshift) at which that occurs and the time evolution of the universe before that epoch.

A. Review of Blackbody Radiation

The spectrum of radiation from a black body of temperature T is given by

$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp[h\nu/kT]-1}. \qquad (10.1)$$

The units are erg s^{-1} cm^{-2} ster^{-1}. The total specific intensity integrated over frequency is

$$I = \int_0^\infty I_\nu d\nu = \frac{\sigma}{\pi}T^4, \qquad (10.2)$$

where $\sigma = 2\pi^5 k^4/15c^2h^3 = 5.7\times10^{-5}$ is Stephan's constant. The total flux radiated outward from a surface is

$$F = \int_{2\pi} I\cos\theta d\omega = \sigma T^4. \qquad (10.3)$$

The units are erg s^{-1} cm^{-2}. The energy density is

$$u = \int_{4\pi} I\cos\theta d\omega = \frac{4\sigma}{c}T^4. \qquad (10.4)$$

The units are erg cm^{-3}. Radiation pressure is given by $P = \int_{4\pi}\frac{I}{c}\cos^2\theta d\omega = \frac{1}{3}u$. The photon density is given by

$$n_\nu = \frac{I_\nu}{h\nu}; \qquad n = \int_0^\infty n_\nu d\nu \propto \frac{u}{kT}. \qquad (10.5)$$

This last integral for the total photon density cannot be done analytically.

For our purposes it is sufficient to know just how radiation properties vary with redshift. We showed previously that the temperature of black-body radation varies with curvature radius R as $T \propto 1/R \propto (1+z)$. Hence the energy density $u \propto 1/R^4 \propto (1+z)^4$. The photon number density varies as $n \propto 1/R^3 \propto (1+z)^3$ which is the same dependence on z as ordinary matter. From the previous chapter, we find that radiation has an equation state given by $w = 1/3$.

The ratio of radiation density to ordinary matter density is given as follows. The radiation density is $u_0 = 4\sigma T^4/c = 4\times10^{-13}$ erg cm^{-3}, where we have used $T = 2.7$ K. The matter density is $\rho_0 = \Omega_0\rho_{crit} = 1.9\times10^{-29}\Omega h^2$ g cm^{-3}. Then

$u_0/(\rho_0 c^2) = 2.3\times 10^{-5}/(\Omega h^2)$. In the past, $\rho_{\rm matter} \propto (1+z)^3$ while $\rho_{\rm rad} \propto (1+z)^4$. Hence $u_r/(\rho c^2) = 1$ at $z = 4 \times 10^4 \Omega_0 h^2$.

Recall that Ω is not a constant for all time but varies with time in the sense that if Ω deviates from unity at any given time, the deviation increases with increasing time. Conversely, if Ω is not unity now, then by going back in time (or looking at high z) there is a point where Ω was still insensibly different from 1. This crossover redshift can be found by use of Equation (6.11):

$$\Omega = \frac{\Omega_0(1+z)}{1+\Omega_0 z}. \tag{10.6}$$

We find that the critical redshift is given roughly by $z \approx 1/\Omega_0$. Now $\Omega > 0.02$ for sure (this being the amount of matter in visible stars) and $\Omega > 0.25$ possibly (if there is as much dark matter as appears to be in clusters); hence the crossover redshift lies less than 50 for sure and less than 5 likely. The upshot is that when the universe was in the radiation dominated phase, it was insensibly different from one that was critically bound. This has the effect of simplifying the treatment of the universe during the radiation dominated phase.

Equation of Motion

The equation of motion are given by Eq. 8.6:

$$\frac{1}{2}\dot{R}^2 + \Phi = \frac{1}{2}\dot{R}^2 - \frac{4}{3}\pi G\rho R^2 = 0. \tag{10.7}$$

Integration of Eq. (10.7) gives

$$R^2 = 2\sqrt{\frac{8\pi G\alpha}{3}}t = \sqrt{\frac{\alpha}{\rho}}. \tag{10.8}$$

Hence

$$t = \sqrt{\frac{3}{32\pi G\rho}} = \frac{2.3}{T_{10}^2}, \tag{10.9}$$

where T_{10} is the temperature of the universe at any epoch in units of 10^{10} K. For example, if $\Omega_0 h^2 = 1$, then the transition between a radiation-dominated and matter-dominated universe happens at $z = 10^4$, where $T_{\rm BB} = 10^5$ K and the age of the universe is 625 years; if $\Omega_0 h^2 = 1/4$, then $z = 10^4$. $T_{\rm BB} = 2.7 \times 10^4$ K, and $t = 10^4$ years. These numbers are approximate only, as they ignore the contribution of neutrinos.

For completeness, note that the equation equivalent to Eq. (10.9) for the matter dominated era is

$$t = \frac{1}{\sqrt{6\pi G\rho_m}}. \tag{10.10}$$

The ratio of baryon number density to photon number density is approximately

$$\frac{n_B}{n_\gamma} = \eta = \frac{\Omega_B \rho_{crit}/\mu}{\rho_r/kT} \approx 10^{-8}\Omega_B h^2. \tag{10.11}$$

More careful computations of the photon number density give 2.8×10^{-8} for the right hand side. Here, μ is the mass of a baryon (1.6×10^{-24} g) and Ω_B is the contribution of baryons alone to the closure density of the universe. This ratio has been nearly constant since very early phases of the universe.

Recombination

At early times the universe was sufficiently hot that matter was almost completely ionized. Today the universe is extremely cold, so any matter not collected into stars is almost completely neutral. Hence there was an epoch when ionized matter (primarily hydrogen and helium) recombined. This epoch is quite important for observations: in the ionized state, the universe is essentially opaque (*i.e.*, the mean free path of a photon is much smaller than the size of the universe, primarily due to Compton scattering off free electrons) while in the neutral state, the universe is essentially transparent. The recombination epoch sets the size of the "visible" universe in the sense that we can observe the universe only to the redshift that corresponds to the recombination era; it is at this epoch that photons in the microwave background radiation last scattered off free electrons before becoming essentially free-streaming, and it is at this epoch that fluctuations in the microwave background temperature were generated. By coincidence, the epoch of recombination is very close to the epoch at which the universe made the transition from radiation-dominated to matter-dominated, with recombination occurring slightly later.

The temperature at which recombination occurs is determined primarily by the atomic physics of the hydrogen atom. To calculate that temperature, it is easiest to borrow the Saha equation from the physics of stellar interiors. The Saha equation gives the ratio of ionized to neutral species provided that they are in thermodynamic equilibrium. Let n_e be the density of electron, n_p be the density of protons, and n_H be the density of neutral hydrogen atoms. Then

$$\frac{n_e n_p}{n_H} = \frac{(2\pi kT)^{3/2}}{h^3} \left(\frac{m_e m_p}{m_H}\right)^{3/2} \exp\left[-\frac{E_R}{kT}\right], \tag{10.12}$$

where m_e, m_p, and m_H are the masses of protons, electrons, and hydrogen atoms respectively, and $E_R = 13.6$ ev is the ionization energy of hydrogen. This equation can be cast into a dimensionless form by introducing variables X_e, X_p, and X_h which we define to be the ratios n/n_B of the various species, with n_B=number density of baryons. We have:

$$n_\gamma = \frac{0.37aT^4}{kT}, \tag{10.12}$$

$$n_B = 2.8 \times 120^{-8}\Omega_B h^2 n_\gamma, \tag{10.13}$$

where n_γ is the number density of photons. Substituting these equations into the Saha equation, we find

$$\frac{X_e X_p}{X_H} = \frac{8.8 \times 10^{13}}{\Omega_b h^2} \left(\frac{E_R}{kT}\right)^{3/2} \exp\left[-\frac{E_R}{kT}\right]. \tag{10.14}$$

Taking $\Omega_B h^2 \approx 0.02$ and $X_e X_p / X_H = 0.5$, we find that $E_R/kT \approx 42$ or $T \approx 3700$ K. This temperature corresponds to a redshift of $z \approx 1400$. The actual temperature depends only logarithmically on the uncertain parameter $\Omega_B h^2$. For plausible values of $\Omega_0 h^2$, recombination occurs after the universe becomes matter dominated. The age of the universe at recombination is

$$t = \frac{1}{\sqrt{6\pi G\rho}} = \frac{130,000 \text{ years}}{\sqrt{\Omega_0 h^2}}. \tag{10.15}$$

One last detail to check is that the universe is indeed in thermodynamic equilibrium. The relevant quantity to check is the recombination rate. Once again, from atomic physics, we find that the recombination rate is given (approximately) by

$$\frac{dn_H}{dt} = n_e n_p \alpha = n_e n_p \times 2.6 \times 10^{-13} \left(\frac{10^4}{T}\right)^{1/2}. \tag{10.16}$$

At $T = 3700$ K and $n_p \approx n_B \approx 5.8 \times 10^{-7} T^3 \Omega_B h^2$ we find that the time scales are

$$t \approx \frac{1}{n_p \alpha} = \frac{2.7}{\Omega_B h^2} \text{ years} \approx 138 \text{ years}. \tag{10.17}$$

Hence the universe is indeed in equilibrium at all times.